Indian Statistical Institute, Bangalore Centre

B.Math (Hons.) II Year, First Semester

Mid-Sem Examination

Analysis III

Time: 3 Hours September 10, 2012 Instructor: T.S.S.R.K. Rao

Total Marks : 5x8=40

Give complete and detailed answers, do not just quote a result from which a solution may follow.

1. Show from the first principles, $f : [0,1] \times [0,1] \to \mathbb{R}$ defined by $f(x,y) = x^2 + y^2$ is integrable.

2. Let I be the unit cube in $\mathbb{R}^3$. Let $f : I \to \mathbb{R}$ be a bounded function such that $f(x,y,z) = 0$ for $x, y, z \in (0,1)$. Show that f is integrable and $I^{\iiint f=0}$.

3. Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Let R be any rectangle containing D. Let $f : R \to \mathbb{R}$ be defined by

$$\begin{aligned} f(x,y) &= 1 \text{ if } x^2 + y^2 \leq 1 \\ &= 0 \text{ if } x^2 + y^2 > 1. \end{aligned}$$

   For any partition

$$P = \left\{ \begin{array}{r} (x_i, y_j) : 0 \leq i \leq n \\ 0 \leq j \leq k \end{array} \right\}$$

   of the rectangle, show that

$$U(P,f) - L(P,f) = \sum_{(i,j) \in S} \sum (x_i - x_{i-1})(y_j - y_{j-1})$$

   where $S = \{(i,j) : [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ has a non-empty intersection with D as well as with $R \backslash D\}$.

4. Let $D = \left\{(x,y) \in \mathbb{R}^2 : -2 \leq x \leq 2 \text{ and } 0 \leq y \leq \sqrt{\frac{(4-x^2)}{2}}\right\}$. Let $f : D \to \mathbb{R}$ be defined by $f(x,y) = y$. Show that f is integrable and compute $\int\int_D f$.

5. Define the notion of a set of content zero in $\mathbb{R}^3$. Show that $D = \{(x, y, \sin(x+y)) : 0 \leq x, y \leq \pi\}$ is a set of content zero.